



$$aNl[Nf, Q^2, L2, \phi] \equiv a(Q^2 = Q^2 \times e^{i\phi}; N_f = N_f; L2 = \Lambda_{N_f}^2; N\text{-loop}; \overline{\text{MS}}), \quad (49)$$

$$aNl_{\text{glob}}[Nf, Q^2, L23, \phi] \equiv a^{(\text{glob.})}(Q^2 = Q^2 \times e^{i\phi}; L23 = \Lambda_3^2; N\text{-loop}; \overline{\text{MS}}). \quad (50)$$

• `trNl[Nf, Nu, k, sig, L2]`: [pQCD](#)

general: it computes the N -loop spectral density including possibly powers of the logarithmic coupling, $\rho_{\nu, k}^{(N)}(\sigma, N_f) = \text{Im}[a(Q^2)^\nu \ln^k(a(Q^2))]_{Q^2 = -\sigma - i\epsilon}$;

input: the number of active flavors $N_f = N_f$; the power index $Nu = \nu$ and the logarithmic power index $k = k$; the squared momentum argument $\text{sig} = \sigma$; the squared $\overline{\text{MS}}$ Lambda QCD parameter $L2 = \Lambda_{N_f}^2$ (all scales in GeV^2);

output: $\rho_{\nu, k}^{(N)}$;

example: In order to compute the value of the three-loop spectral density, at $\sigma = 1.5 \text{ GeV}^2$ and $N_f = 3$, and with $\Lambda_{N_f}^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $\rho_{0.5, 0}^{(3)}(1.5, 3) = 0.104393$, one has to use the command `tr3l[3, 0.5, 0, 1.5, 0.1]`.

• `trNl[Nf, ν , k, σ , $\Lambda_{N_f}^2$]` returns N -loop perturbative spectral density $\rho_{\nu, k}^{(N)}(\sigma; N_f) = \text{Im}[a^\nu \ln^k a]_{Q^2 = -\sigma - i\epsilon}$ ($N = 1, 2, 3, 4$) of real power ν and logarithmic power k at σ and at fixed number of active quark flavors N_f :

$$\text{trNl}[Nf, \nu, k, \sigma, L2] = \rho_{\nu, k}^{(N)}[\sigma; N_f = N_f; L2 = \Lambda_{N_f}^2] \quad (51)$$

$(\nu \in \mathcal{R}; k = 0, 1, \dots; N = 1, 2, 3, 4; N_f = 3, 4, 5, 6).$

• `trNlglob[Nu, k, sig, L2nf3]`:

general: it computes the N -loop global spectral density incorporating the powers of the logarithmic coupling

$$\rho_{\nu, k}^{(N)\text{glob.}}(\sigma, N_f) = \text{Im}[a^{(\text{glob.})}(Q^2)^\nu \ln^k(a^{(\text{glob.})}(Q^2))]_{Q^2 = -\sigma - i\epsilon};$$

input: the power index $Nu = \nu$ and the logarithmic power index $k = k$; the squared momentum argument $\text{sig} = \sigma$; the squared $\overline{\text{MS}}$ Lambda QCD parameter at $N_f = 3$ (at the corresponding N -loop) $L2nf3 = \Lambda_3^2$ (all scales are in GeV^2);

output: $\rho_{\nu, k}^{(N)\text{glob.}}$;

example: In order to compute the value of the three-loop global spectral density at $\sigma = 1.5 \text{ GeV}^2$ and with $\Lambda_3^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $\rho_{0.5, 0}^{(3)\text{glob.}}(1.5, 3) = 0.104393$, one has to use the command `tr3lglob[0.5, 0, 1.5, 0.1]`.

• `trNlglob[ν , k, σ , Λ_3^2]` returns N -loop global perturbative spectral density $\rho_{\nu, k}^{(N)\text{glob.}}(\sigma; N_f)$ ($N = 1, 2, 3, 4$) of real power ν and logarithmic power k at σ , and with Λ_3 being the QCD $N_f = 3$ scale:

$$\text{trNlglob}[\nu, k, \sigma, L23] = \rho_{\nu, k}^{(N)\text{glob.}}[\sigma; L23 = \Lambda_3^2], \quad (N = 1, 2, 3, 4). \quad (52)$$

- **AFAPT $N1$ [$N_f, N_u, k, Q2, L2, Fi$]:**

FAPT

general: it computes the N -loop coupling in $FAPT_{N_f}$ incorporating the analytization of powers of the logarithmic coupling

$$\mathcal{A}_{\nu,k}^{(FAPT,N)}(Q^2, N_f) = (a^\nu(Q^2) \ln^k a(Q^2))_{\text{an.FAPT}} \text{ in the Euclidean domain;}$$

input: the number of active flavors $N_f = N_f$; the power index $N_u = \nu$ and the logarithmic power index $k = k$; the squared momentum argument $Q2 = |Q^2|$; the squared $\overline{\text{MS}}$ Lambda QCD parameter $L2 = \overline{\Lambda}_{N_f}^2$; the phase of the complex $Q^2 = |Q^2|e^{i\phi}$, i.e., $Fi = \phi$ (in radians); all scales are in GeV^2 ;

output: $\mathcal{A}_{\nu,k}^{(FAPT,N)}$;

example: In order to compute the value of the three-loop FAPT coupling \mathcal{A}_ν at $Q^2 = 1.5 \text{ GeV}^2$, with $\nu = 0.5$, $N_f = 3$ and $\overline{\Lambda}_3^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $\mathcal{A}_{0.5,0}^{(FAPT,3)}(1.5, 3) = 0.324597$, one has to use the command
AFAPT31[3, 0.5, 0, 1.5, 0.1, 0].

- **AFAPT $N1$ [$N_f, \nu, k, |Q^2|, \Lambda^2, \phi$]** returns N -loop ($N = 1, 2, 3, 4$) analytic FAPT coupling $\mathcal{A}_{\nu,k}^{(FAPT,N)}(Q^2, N_f) = (a^\nu(Q^2) \ln^k a(Q^2))_{\text{an.FAPT}}$, of real power ν and logarithmic power k at fixed number of active quark flavors N_f , in the Euclidean domain [$Q^2 = |Q^2| \exp(i\phi) \in \mathcal{C}$ and $Q^2 \not\prec 0$], with Q^2 in units of GeV^2 and ϕ in radians

$$\begin{aligned} \text{AFAPT}N1[N_f, \nu, k, Q2, L2, \phi] &= \\ &= \mathcal{A}_{\nu,k}^{(FAPT,N)}[Q2 = |Q^2|, \phi = \arg(Q^2); N_f = N_f; L2 = \overline{\Lambda}_{N_f}^2] \\ &\quad (N = 1, 2, 3, 4; N_f = 3, 4, 5, 6). \end{aligned} \quad (53)$$

- **AFAPT $N1$ glob[$N_u, k, Q2, L2nf3, Fi$]:**

example: In order to compute the value of the three-loop FAPT coupling \mathcal{A}_ν at $Q^2 = 1.5 \text{ GeV}^2$, with $\nu = 0.5$ and $\overline{\Lambda}_3^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $\mathcal{A}_{0.5,0}^{(FAPT,3)\text{glob.}}(1.5) = 0.333458$, one has to use the command
AFAPT31glob[0.5, 0, 1.5, 0.1, 0].

general: it computes the N -loop global FAPT coupling $\mathcal{A}_{\nu,k}^{(FAPT,N)\text{glob.}}(Q^2)$ (Euclid.)

input: the power index $N_u = \nu$ and the logarithmic power index $k = k$; the squared momentum argument $Q2 = |Q^2|$; the squared $\overline{\text{MS}}$ Lambda QCD parameter at $N_f = 3$ $L2nf3 = \overline{\Lambda}_3^2$; the phase of the complex $Q^2 = |Q^2|e^{i\phi}$, i.e., $Fi = \phi$ (in radians); all scales are in GeV^2 ;

output: $\mathcal{A}_{\nu,k}^{(FAPT,N)\text{glob.}}$; $Q^2 = 1.5 \text{ GeV}^2$, with $\nu = 0.5$ and $\overline{\Lambda}_3^2 = 0.1 \text{ GeV}^2$, i.e.,

$$\mathcal{A}_{0.5,0}^{(FAPT,3)\text{glob.}}(1.5) = 0.333458, \text{ one has to use the command}$$

$$\text{AFAPT31glob}[0.5, 0, 1.5, 0.1, 0].$$

- In the global FAPT case **AFAPT $N1$ glob[$\nu, k, |Q^2|, \Lambda_3^2, \phi$]** returns N -loop analytic FAPT coupling $\mathcal{A}_{\nu,k}^{(FAPT,N)\text{glob.}}(Q^2)$, of real power ν and logarithmic power k , in ν, k the Euclidean domain,

(54)

$$\begin{aligned} \text{AFAPT}N1\text{glob}[\nu, k, Q2, L23, \phi] &= \mathcal{A}_{\nu,k}^{(FAPT,N)\text{glob.}}[Q2 = |Q^2|, \phi = \arg(Q^2); \\ L23 &= \overline{\Lambda}_3^2] \quad (N = 1, 2, 3, 4). \end{aligned}$$

- **AMPTN1** [**Nf**, **Nu**, **Q2**, **M2**, **L2MPT**] :

MPT

general: it computes the N -loop coupling $\mathcal{A}_\nu^{(\text{MPT},N)}(Q^2, m_{\text{gl}}^2, N_f)$ in MPT_{N_f} in the Euclidean domain;

input: the number of active flavors $\text{Nf}=N_f$; the index $\text{Nu}=\nu$ ($0 < \nu < 5$); the squared momentum argument $\text{Q2}=Q^2$ (complex in general); the squared $\overline{\text{MS}}$ Lambda QCD parameter $\text{L2MPT}=\overline{\Lambda}_{N_f}^2$; the effective mass parameter $\text{M2}=m_{\text{gl}}^2$ (all scales in GeV^2);

output: $\mathcal{A}_\nu^{(\text{MPT},N)}$;

example: In order to compute the value of the three-loop MPT coupling \mathcal{A}_ν with $N_f = 3$, $\nu = 1.4$, at $Q^2 = 0.5 \text{ GeV}^2$, with $m_{\text{gl}}^2 = 0.7 \text{ GeV}^2$, and $\overline{\Lambda}_3^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $\mathcal{A}_{1.4}^{(\text{MPT},3)}(0.5, 0.7, 3) = 0.0514469$, one has to use the command **AMPT31** [**3**, **1.4**, **0.5**, **0.7**, **0.1**].

- **AMPTN1** [$N_f, \nu, Q^2, m_{\text{gl}}^2, \overline{\Lambda}_{N_f}^2$] returns N -loop ($N = 1, 2, 3, 4$) analytic MPT coupling $\mathcal{A}_\nu^{(\text{MPT},N)}(Q^2, m_{\text{gl}}^2, N_f)$, of fractional power ν ($0 < \nu < 5$) and at fixed number of active quark flavors N_f , with Q^2 in the Euclidean domain ($Q^2 \in \mathcal{C}$ and $Q^2 \not\prec 0$)

$$\begin{aligned} \text{AMPTN1}[N_f, \nu, Q^2, M2, L2] &= \\ &= \mathcal{A}_\nu^{(\text{MPT},N)} [Q2 = Q^2 \in \mathcal{C}; Nf = N_f; M2 = m_{\text{gl}}^2; L2 = \overline{\Lambda}_{N_f}^2] \\ &\quad (N = 1, 2, 3, 4; Nf = 3, 4, 5, 6); 0 < \nu < 5). \end{aligned} \quad (58)$$

- **tAMPTN1** [**Nf**, **n**, **nu**, **Q2**, **M2**, **L2MPT**] :

general: it computes the coupling $\tilde{\mathcal{A}}_{n+\nu}^{(\text{MPT},N)}(Q^2, m_{\text{gl}}^2, N_f)$, the generalized logarithmic derivative with index $n + \nu$, in MPT_{N_f} in the Euclidean domain;

input: the number of active flavors $\text{Nf} = N_f$; the integer index n ($= 0, 1, 2, 3, 4$) and the noninteger index $\text{nu}=\nu$ ($0 \leq \nu < 1$); the squared momentum argument $\text{Q2}=Q^2$ (complex in general); the effective mass parameter $\text{M2}=m_{\text{gl}}^2$; the squared $\overline{\text{MS}}$ Lambda QCD parameter $\text{L2MPT}=\overline{\Lambda}_{N_f}^2$ (all scales in GeV^2); all scales are in GeV^2 ;

output: $\tilde{\mathcal{A}}_{n+\nu}^{(\text{MPT},N)}$;

example: In order to compute the value of the three-loop MPT coupling $\tilde{\mathcal{A}}_{n+\nu}$ with $N_f = 3$, with $n = 1$ and $\nu = 0.4$, at $Q^2 = 0.5 \text{ GeV}^2$, with $m_{\text{gl}}^2 = 0.7 \text{ GeV}^2$, and $\overline{\Lambda}_3^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $\tilde{\mathcal{A}}_{1.4}^{(\text{MPT},3)}(0.5, 0.7, 3) = 0.0528178$, one has to use the command **tAMPT31** [**3**, **1**, **0.4**, **0.5**, **0.7**, **0.1**].

- **tAMPTN1** [$N_f, n, \nu, Q^2, m_{\text{gl}}^2, \overline{\Lambda}_{N_f}^2$] returns N -loop ($N = 1, 2, 3, 4$) analytic MPT coupling $\tilde{\mathcal{A}}_{n+\nu}^{(\text{MPT},N)}(Q^2, m_{\text{gl}}^2, N_f)$, the generalized logarithmic derivative with index $n + \nu$ ($n = 0, 1, 2, 3, 4; 0 \leq \nu < 1$), at fixed number of active quark flavors N_f , with Q^2 in the Euclidean domain ($Q^2 \in \mathcal{C}$ and $Q^2 \not\prec 0$)

$$\begin{aligned} \text{tAMPTN1}[N_f, n, \nu, Q^2, M2, L2] &= \\ &= \tilde{\mathcal{A}}_{n+\nu}^{(\text{MPT},N)} [Q2 = Q^2 \in \mathcal{C}; Nf = N_f; M2 = m_{\text{gl}}^2; L2 = \overline{\Lambda}_{N_f}^2] \\ &\quad (N = 1, 2, 3, 4; Nf = 3, 4, 5, 6); n = 0, 1, 2, 3, 4; 0 \leq \nu < 1). \end{aligned} \quad (57)$$

- `tA2d[Nf,nu,Q2,Fi]`:

general: it computes coupling $\tilde{\mathcal{A}}_{\text{nu}}^{(2\delta)}(Q^2, N_f)$ in $2\delta\text{anQCD}_{N_f}$, the generalized logarithmic derivative with index `nu`, in the Euclidean domain;

input: the number of active flavors $N_f = N_f$; the index $\nu = \text{nu}$ ($\text{nu} > -1$ and real); the squared momentum argument $Q^2 = |Q^2|$ (in GeV^2); $\text{Fi} = \phi$ is the phase of the complex $Q^2 = |Q^2|e^{i\phi}$ (in radians);

output: $\tilde{\mathcal{A}}_{\nu}^{(2\delta, N)}$;

example: In order to compute the value of $\tilde{\mathcal{A}}_{\nu}$ at $Q^2 = 0.5 \text{ GeV}^2$, with $\text{nu} = 1.4$, and $N_f = 3$, i.e., the coupling $\tilde{\mathcal{A}}_{1.4}^{(2\delta, 3)}(0.5) = 0.0827052$, one has to use the command `tA2d[3, 1.4, 0.5, 0]`.

- `tA2d[Nf, ν, |Q2|, φ]` returns analytic $2\delta\text{anQCD}$ coupling $\tilde{\mathcal{A}}_{\nu}^{(2\delta)}(Q^2, N_f)$, the generalized logarithmic derivative with index ν ($\nu > -1$ and real, in general non-integer), at fixed number of active quark flavors N_f , in the Euclidean domain $Q^2 = |Q^2| \exp(i\phi) \in \mathcal{C} \setminus [-M_{\text{thr}}^2, -\infty)$ where $M_{\text{thr}}^2 = M_2^2 (= s_2 s_0[N_f] LL2[N_f])$

$$\text{tA2d}[Nf, \nu, Q2, \phi] = \tilde{\mathcal{A}}_{\nu}^{(2\delta)}[Q2 = |Q^2|, \phi = \arg(Q^2); Nf = N_f],$$

$$(Nf = 3, 4, 5, 6; \nu > -1). \quad (55)$$

- `A2dN1[Nf,n,nu,Q2,Fi]`:

general: it computes N -loop coupling $\mathcal{A}_{\text{nu}+\text{n}}^{(2\delta)}(Q^2, N_f)$ in $2\delta\text{anQCD}_{N_f}$ in the Euclidean domain;

input: the number of active flavors $N_f = N_f$; the indices \mathbf{n} (\mathbf{n} is nonnegative integer) and nu ($\text{nu} > -1$ and real); the squared momentum argument $Q^2 = |Q^2|$ (in GeV^2), $\text{Fi} = \phi$ is the phase of the complex $Q^2 = |Q^2|e^{i\phi}$ (in radians); see also Eq. (28), with $\nu_0 \mapsto \text{nu}$ and $n \mapsto \mathbf{n}$;

output: $\mathcal{A}_{\nu+n}^{(2\delta, N)}$;

example: In order to compute the value of the “three-loop” $2\delta\text{anQCD}$ coupling \mathcal{A}_{ν} at $Q^2 = 0.5 \text{ GeV}^2$, with $\text{nu} = 0.4$, $\mathbf{n} = 1$ and $N_f = 3$, i.e., the coupling $\mathcal{A}_{1.4}^{(2\delta, 3)}(0.5) = 0.0745576$, one has to use the command `A2d31[3, 1, 0.4, 0.5, 0]`.

- `A2dN1[Nf, n, ν, |Q2|, φ]` returns N -loop analytic $2\delta\text{anQCD}$ coupling $\mathcal{A}_{\nu+n}^{(2\delta)}(Q^2, N_f)$, of fractional power $n+\nu$ ($\nu > -1$ and real; $n = 0, 1, \dots, N-1$) at fixed number of active quark flavors N_f , in the Euclidean domain $Q^2 = |Q^2| \exp(i\phi) \in \mathcal{C} \setminus [-M_{\text{thr}}^2, -\infty)$ where $M_{\text{thr}}^2 = M_2^2$, used for the $N^{N-1}\text{LO}$ truncation approach [cf. Eqs. (26)-(30), in particular Eq. (28) with $\nu \mapsto \nu_0$]

$$\text{A2dN1}[Nf, n, \nu, Q2, \phi] = \mathcal{A}_{\nu+n}^{(2\delta)}[Q2 = |Q^2|, \phi = \arg(Q^2); Nf = N_f],$$

$$(N = 1, 2, 3, 4, 5; Nf = 3, 4, 5, 6; n = 0, 1, \dots, N-1). \quad (56)$$